Magnitude-Redshift relation in the Brans-Dicke theory with a variable cosmological term

2009 February

Theories of Particle Physics, Nuclear Physics and Astrophysics/Astronomy,
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Abstract

We have investigated the magnitude-redshift relation in the Brans-Dicke theory with both a variable and constant cosmological terms.

Cosmological models with a cosmological term are tightly constrained by the magnitude redshift relation of SNIa observations. This is because the cosmological term affects significantly to the cosmic expansion rate of the universe at the low redshifts of $z$.

For our investigations, we use observations of SNIa, redshift $z$ in the range of $0.01 < z < 2$. First we examine the contribution of the matter and a variable cosmological term for the flat universe in the Brans-Dicke model with a variable cosmological term ($BD\Lambda$). It is noticed that the matter contribution is very high in this model at the present epoch and variable cosmological term is small compared with matter density. Next, magnitude-redshift relation is investigated for this model. This model is inconsistent with the present accelerating universe which is equivalent as the matter dominant universe in the Friedmann model where $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$.

Therefore as the next approach $BD\Lambda$ model with the constant cosmological term has been investigated. We fix the energy density parameter of the constant cosmological term as 0.7. The total density of the cosmological term is more dominant than the density of the matter. By using $m - z$ relation, it implies that the accelerating universe has the same energy density parameters as the Friedmann model where $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ with the $\chi^2 = 196$ value; here $\Omega_m$ is the energy density parameter of the matter and $\Omega_\Lambda$ is the energy density parameter of the cosmological term. We find that this model is also parameter independent. We can not constrain cosmological parameters which is inherent in the $BD\Lambda$ theory.
1 Introduction

General theory of relativity is the most successful gravitational theory being almost universally accepted. So far, cosmological observations which are almost in case of weak gravitational field agree with the predictions of general relativity. And also it is the theory that has been related to many interesting phenomena in cosmology.

Even though the cosmological standard models based on Einstein theory succeed to explain some theoretical points of cosmology related to the observations, it has been remaining some puzzles in physics like, horizon, flatness and monopole problem etc. Accelerating expansion observations of the universe also can not be explained by the standard model though it has a strong evidence of the existence of the cosmological term.

Using distant standard candles where objects are known in absolute luminosity such as type Ia Supernovae (SNIa), it is possible to start seeing the varied effects of the universe’s expansion history. Such cosmological observations have indicated that the expansion of the universe is accelerating during the present redshift times [1, 2]. This causes to change the way of thinking about the universe.

Conventionally, the world of particle physics and cosmology would be unified in the early universe. Particle physics provided some needed sources of energy density during the period, leading to the process like inflation [3]. From the point of particle physics, cosmological constant naturally arises as an energy density of a vacuum. If the cosmological constant originated from the vacuum energy density, energy scale of cosmological term $\Lambda$ should be much larger than the Hubble constant ($H_0$). This is the ’’cosmological
constant problem” [4].

Observationally we know, \( \Lambda \approx H_0^2 = (2.13h \times 10^{-42}\text{GeV})^2 \). This corresponds to the critical density,

\[
\rho_\Lambda = \frac{\Lambda m_{pl}^2}{8\pi} = \frac{3H_0^2}{8\pi G} \sim 10^{-47}\text{GeV}^4,
\]

where the plank mass \( m_{pl} \) is written as,

\[
m_{pl} = \sqrt{\frac{hc}{2\pi G}},
\]

with the velocity of the light \( c = 1 \), and \( h \) is the Planck constant. Here, \( G \) is the gravitational constant. As well vacuum energy density in the plank scale can be evaluated as [3],

\[
\rho_{\text{vac}} \approx 10^{74}\text{GeV}^4.
\]

It seems that it is about \( 10^{121} \) orders of magnitude larger than the present observed value. This requires a fine-tuning to adjust \( \rho_\Lambda \) to the present energy density of the universe.

To explain these puzzles in cosmology, new modified theories beyond the standard model are needed. It is worthwhile to suggest a cosmological term as a function of time, which is decreasing from large value at the early universe to the the present value. Cosmological models with cosmological terms are tightly constrained by the magnitude-redshift relation derived from SNIa observations. This is because, the cosmological term affects significantly to the cosmic expansion rate of the universe at the low red shifts.

The most important class of deviant theories from general relativity are scalar-tensor gravity theories, of which the Brans-Dicke theory [5, 6] is the simplest and best-studied generalization of general relativity. Referring the theories developed by [5, 7], scalar-tensor theories are proposed by introducing the cosmological term \( \Lambda \) which as a function of a scalar field \( \phi \). In this framework [8], they derived a concrete functional form for the \( \Lambda \). This theory leads to a variation in the Newtonian gravitation "constant" \( G \), and introduces a new coupling constant \( \omega \), with general relativity recovered in the limit \( 1/\omega \rightarrow 0 \). This theory contains some unknown physical quantities which depend on an initial scalar field; the scalar field determines the ages of the universe, dark energy, and
the gravitational constant.

Brans-Dicke model with a variable cosmological term has been investigated from Big Bang nucleosynthesis [9, 10, 11] for the early universe. However, we want to study ”How this model works at the present epoch?”. Therefore to investigate this model, we adopt the magnitude-redshift \((m - z)\) relation of SNIa observations.

In the second chapter, magnitude-redshift relation is investigated in the Friedmann model with SNIa observations [1]. Related with the cosmological constant problem, beyond the standard model the Brans-Dicke model with a variable cosmological term is introduced in the chapter three. In the fourth chapter, \(m - z\) relation is investigated in the Brans-Dicke model with the variable cosmological term. Furthermore, \(m - z\) relation in particular for the flat universe of the \(BD\Lambda\) model is studied with and without a cosmological constant. In the chapter five, summary and discussion are given and future works are also added.
2 Standard Cosmology

2.1 Outline of the Friedmann model

Albert Einstein is the person who introduced the constant cosmological term to the theory of general relativity in 1917. The Scientific establishment at the start of the twentieth century was confident that the universe was static and eternal. Therefore, Einstein added the cosmological constant to the general theory of relativity to give a rice to an anti-gravitational effect to stop the universe collapsing. This was fit with general views of a static and eternal universe.

Alexander Friedmann explained the expanding universe using general relativity without cosmological term. Georges Lemaitre got the analytical solution for the Einstein equations with a cosmological term, designating the accelerating universe. Einstein discarded it because of the discovery of the expanding universe by Edwin Hubble in 1929.

2.2 Magnitude-redshift relation

The cosmic distance measures depend sensitively on the spatial curvature and the expansion dynamics of the models and consequently on the present density of the various energy components and their equation of state. For this reason, the magnitude-redshift \((m - z)\) relation for distant standard candles can be proposed as a potential test for cosmological models and plays a crucial role in determining cosmological parameters. Distance can be measured through the luminosity of a stellar object.

Redshift is often used to describe the evolution of the universe. This is because the light emitted by the stellar objects becomes redshifted due to the expansion of the universe.
For the homogeneous and isotropic universe, Robertson-Walker metric can be written as,

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\psi^2) \right] \]  

(2.1)

where \( a(t) \) is the scale factor with cosmic time \( t \). The coordinates \( r, \theta, \psi \) are known as comoving coordinates. Here we set the velocity of the light \( c = 1 \).

The wavelength \( \lambda \) increases proportionally to the scale factor \( a \), whose effects can be quantified by the redshift \( z \) [3], as,

\[ 1 + z = \frac{\lambda_0}{\lambda} = \frac{a_0}{a} \]  

(2.2)

where the subscript 0 indicates denotes the quantities given at the present epoch. One way of defining a distance is through the luminosity of a stellar object. The distance \( d_L \) is known as the luminosity distance, plays a very important role in astronomy including the Supernovae observations.

In Minkowski space time the absolute luminosity \( L_s \) of the source and the energy flux \( F \) at a distance \( d \) is related through \( F = L_s/(4\pi d^2) \). By generalizing this to an expanding universe, the luminosity distance, \( d_L \), is defined as,

\[ d_L^2 \equiv \frac{L_s}{4\pi F}. \]  

(2.3)

Let us consider an object with absolute luminosity \( L_s \) located at a coordinate distance \( r_l \) from an observer at \( r = 0 \) [see the metric Eq. (2.1)]. The energy of light emitted from the object with time interval \( \Delta t_1 \) is denoted as \( \Delta E_1 \), whereas the energy which reaches at the sphere with radius \( r_l \) is written as \( \Delta E_0 \). We note that \( \Delta E_1 \) and \( \Delta E_0 \) are proportional to the frequencies of light at \( r = r_l \) and \( r = 0 \), respectively, i.e., \( \Delta E_1 \propto \nu_1 \) and \( \Delta E_0 \propto \nu_0 \). The luminosity \( L_s \) and \( L_0 \) are given by

\[ L_s = \frac{\Delta E_1}{\Delta t_1}, \quad L_0 = \frac{\Delta E_0}{\Delta t_0}. \]  

(2.4)

The speed of light is given by \( c = \nu_1 \lambda_1 = \nu_0 \lambda_0 \), where \( \lambda_1 \) and \( \lambda_0 \) are the wavelengths at \( r = r_l \) and \( r = 0 \). Then from Eq. (2.2) we find
\[ \lambda_0 = \frac{\nu_1}{\lambda_1}, \quad \frac{\Delta t_0}{\nu_0} = \frac{\Delta E_1}{\Delta E_0} = 1 + z, \quad (2.5) \]

where we have also used \( \nu_0 \Delta t_0 = \nu_1 \Delta t_1 \). Combining Eq. (2.5) with Eq. (2.5), we obtain

\[ L_\ast = L_0 (1 + z)^2. \quad (2.6) \]

The light traveling along the radial direction \( r \) satisfies the geodesic equation,

\[ ds^2 = -dt^2 + \frac{a^2 dr^2}{1 - kr^2} = 0, \quad (2.7) \]

We then obtain

\[ \int_0^{r_0} \frac{dt}{a(t)} = \int_0^{r_i} \frac{dr}{\sqrt{(1 - kr^2)}}. \quad (2.8) \]

Curvature constant \( k \) is related with the spacial geometry of the universe. The universe is flat if \( k = 0 \), closed if \( k > 0 \), and open if \( k < 0 \).

Then Eq. (2.8) can be integrated as,

\[ \int_0^{r_0} \frac{dt}{a(t)} = \begin{cases} 
  k^{-1/2} \sin^{-1}\left(\sqrt{kr_i}\right) & k > 0 \\
  r_i & k = 0 \\
  |k|^{-1/2} \sinh^{-1}\left(\sqrt{|k|} r_i\right) & k < 0 
\end{cases} \]

From the metric Eq. (2.7) we find that the area of the sphere at \( t = t_0 \) is given by \( S = 4\pi a_0^2 r_i^2 \). Hence the observed energy flux is

\[ F = \frac{L_0}{4\pi a_0^2 r_i^2}. \quad (2.9) \]

Substituting Eq (2.8) and Eq. (2.8), we obtain the luminosity distance in an expanding universe:

\[ d_L = (1 + z) a_0 r_i \quad (2.10) \]

The apparent magnitude \( m \) of the source with an absolute magnitude \( M \) is related to the luminosity distance \( d_L \) via the relation

\[ m = M + 5 \log_{10} \frac{d_L}{10 \text{[pc]}}, \quad (2.11) \]
where absolute magnitude is the apparent magnitude, when the light source is placed at the 10 [pc] (1pc = 3.08568025 × 10^{13}km) distance.

Eq. (2.4) comes from taking the logarithm of Eq. (2.2) by noting that \( m \) and \( M \) are related to the logarithms of \( F \) and \( L_s \), respectively [3]. The numerical factor arise because of conventional definitions of \( m \) and \( M \) in astronomy [3]. If the observer receives light at the present epoch, substituting \( d_L \) from Eq. (2.3) to Eq. (2.4), it is written as,

\[
m = M + 5\log_{10}\left(\frac{1+z}{10} \right),
\]  

where the present scale factor \( a_0 = 1 \).

Defining Hubble parameter \( H = \dot{a}/a \) and \( a = 1/(1+z) \), Eq. (2.8) can be written as,

\[
\int_0^z \frac{dz}{H} = \int_0^{r_l} \frac{dr}{\sqrt{1-kr^2}}.
\]

Now we consider the Friedmann equation. Considering the Einstein Field equation on the metric for a general isotropic homogeneous universe, we could find the solutions for Einstein’s field equation. Friedmann equation also defined as a Robertson-Walker solution to the Einstein’s field equation as,

\[
H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} (\rho_m + \rho_\gamma) - \frac{k^2}{a^2} + \frac{\Lambda}{3},
\]

where each symbol indicates as follows;

- \( H \) : Hubble parameter
- \( a \) : Scale factor
- \( k \) : Curvature constant
- \( \rho_m \) : Energy density of baryon and dark matter
- \( \rho_\gamma \) : Energy density of radiation
- \( \Lambda \) : Cosmological constant

Then, Friedmann equation can be rewritten as,

\[
H^2 = H_0^2 \left( \frac{\Omega_m}{a^3} + \frac{\Omega_\gamma}{a^2} + \frac{\Omega_k}{a^2} + \Omega_\Lambda \right),
\]  

10
where,

\[ \Omega = \frac{\rho}{\rho_{cr}} : \text{Energy density parameter} \]

\[ \Omega_\Lambda = \frac{\Lambda}{3H_0^2} : \text{Energy density parameter of cosmological constant} \]

\[ \Omega_k = -\frac{k}{H_0^2} : \text{Energy density parameter of the curvature} \]

\[ \rho_{cr} = \frac{3H_0^2}{8\pi G} : \text{Critical density for Friedmann model} \]

where \( H_0 \) is for "Hubble constant" at the present epoch.

"Critical density" is the density for which the universe is spatially flat in the absence of a cosmological constant for a given value of a Hubble parameter [12].

Substituting \( a = 1/(1 + z) \) in Eq. (2.11),

\[ H^2 = H_0^2 \left( \Omega_m (1 + z)^3 + \Omega_\gamma (1 + z)^4 + \Omega_k (1 + z)^2 + \Omega_\Lambda \right). \]  \hspace{1cm} (2.16)

Applying the present redshift \( (z = 0) \) and \( H(z = 0) = H_0 \),

\[ \Omega_m + \Omega_\gamma + \Omega_k + \Omega_\Lambda = 1. \]  \hspace{1cm} (2.17)

Substituting the \( H \) from Eq. (2.12) to Eq. (2.9) for each case of \( k \), radial distance \( r_l \) can be calculated. By using Eq. (2.5) apparent magnitude \( m \) also can be calculated.

### 2.2.1 SNIa observations

The time evolution of the cosmic scale factor depends on the composition of mass-energy in the Universe. While the Universe is known to contain a significant amount of ordinary matter, \( \Omega_m \), which decelerates the expansion, its dynamics may also be significantly affected by more exotic forms of energy. Most successful form among those is a possible energy of the vacuum (\( \Omega_\Lambda \)), Einstein’s "cosmological constant,” whose negative pressure would do work to accelerate the expansion [13].
Although it is easy to measure the apparent magnitudes, it is hard to measure the absolute magnitude of the distant objects \([13, 14]\). By the progress of the measuring the Type Ia supernovae as "standard candles" \([15]\), it plays a crucial role of understanding the cosmological effects. Supernovae SNIa in particular all seems to be of nearly uniform intrinsic luminosity. Since of the brightness of SNIa is equivalent in entire galaxy where they appear, it can be also detected at high redshifts.

Therefore measurements of the redshift and apparent brightness of SNIa of known intrinsic brightness can constrain above cosmological parameters and it allows to handle cosmological effects.

Figure 2.1: Hubble Diagram from the Supernovae Cosmology project\([1]\). Recent observations of high redshift type Ia supernovae (SNIa) strongly suggest that the expansion of the Universe is accelerating \([1, 16, 17, 18]\).
2.2.2 Numerical results

Energy density of the radiation today is much less than that in matter. Photons, which are readily detectable, contribute $\Omega_\gamma \sim 5 \times 10^{-5}$, mostly in the 2.73 K cosmic microwave background [13, 14, 19]. Therefore, it is worthwhile to parameterize the present universe by the value of $\Omega_m$ and $\Omega_\Lambda$, with $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$.

![Figure 2.2: Calculated apparent magnitude for a range of cosmological models without $\Omega_k$: $(\Omega_m, \Omega_\Lambda) = (0.0, 1.0)$ on top, (0.5, 0.5) in the second, (1.0, 0.0) in the third and (1.5, -0.5) on bottom](image)

Figure 2.2 and Figure 2.3 represent the model universes and their fate with and without $\Omega_k$ and $\Omega_\Lambda$. Figure 2.2 represents the magnitude-redshift $(m - z)$ relation for $\Omega_k = 0$, where we adopt $H_0 = 71$ [km/sec/Mpc], $M = -19.1$. When the universe is dominant by the matter, in the absence of cosmological term, Eq. (2.11) is rewritten as,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2 \Omega_m}{a^3}. $$

Therefore, we can get following relations,

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^3}.$$
Figure 2.3: Calculated apparent magnitude for a range of cosmological models without \( \Omega_\Lambda \) : \((\Omega_m, \Omega_k) = (-0.5, 1.5)\) on top, \((1.0, 0.0)\) in the second, \((1.5, -0.5)\) in the third and \((2.5, -1.5)\) on bottom.

Figure 2.4: Magnitude-redshift relation with SNIa observations for a range of cosmological models without \( \Omega_k \) : \((\Omega_m, \Omega_\Lambda) = (0.0, 1.0)\) on top, \((0.5, 0.5)\) in the second, \((1.0, 0.0)\) in the third and \((1.5, -0.5)\) on bottom.
Figure 2.5: Magnitude-redshift relation with SNIa observations for a range of cosmological models without $\Omega_\Lambda$: $(\Omega_m, \Omega_k) = (-0.5, 1.5)$ on top, $(1.0, 0.0)$ in the second, $(1.5, -0.5)$ in the third and $(2.5, -1.5)$ on bottom.

\[
a^2 \, da \propto dt
\]

\[
a \propto t^\frac{2}{3}.
\]

It implies the matter dominant universe with the absence of cosmological term is decelerating. Since $\Omega_m$ is multiplied by $(1 + z)^3$, for large $\Omega_m$, $H(z)$ increases with $z$ as seen in Eq. (2.12). Since apparent magnitude $m$ is proportional to $H(z)^{-1}$ at the present epoch, $m$ decreases when the matter increases.

If the universe is dominant by $\Omega_\Lambda$, with the absence of $\Omega_m$, we can get the following relations;

\[
\left( \frac{\dot{a}}{a} \right)^2 = H_0^2 \Omega_\Lambda
\]

\[
\left( \frac{\dot{a}}{a} \right)^2 \propto \text{constant}
\]
\[
\frac{da}{a} \propto dt \\
\frac{a}{e^t} \propto e^t
\]

It has shown that a cosmological constant dominant universe in the absence of matter, scale factor growing exponentially with time. Figure 2.2 shows the results of the universe without \( \Omega_k \).

Figure 2.3 shows the \( m - z \) relation for \( \Omega_\Lambda = 0 \). In this model total density contribute only from matter and curvature densities. When \( \Omega_k \) increases \( \Omega_m \) decreases. Since \( \Omega_k \) is multiplied by the \( (1 + z)^2 \), \( H(z) \) also increases as \( z \). When, \( \Omega_k > 0 \), \( m \) increases compared with \( \Omega_k = 0 \) case. When \( \Omega_k < 0 \), \( H(z) \) increases compared with the case of \( \Omega_k = 0 \) and it implies that \( m \) decreases.

In the Friedmann equation when the universe is dominant by the curvature (empty universe) [13], we can simplify the Friedmann equation for \( \Omega_m = \Omega_\Lambda = 0, \Omega_k = 1 \) as,

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{H_0^2}{a^2} = \text{constant}
\]

This implies that the universe accelerate linearly for curvature dominant universe.

Figure 2.4 and Figure 2.5 represent the \( m - z \) relation with the SN Ia observational data from SCP (Supernovae Cosmology project) and HZT (High-z supernovae search team) [20].

Considering the above two figures it is natural to ask what level of confidence is suite with these parameters. Therefore to find the confidence levels for these parameters, we have done the \( \chi^2 \) fittings for \( \Omega_m \) and \( \Omega_\Lambda \) in the range of \( 0.0 \leq \Omega_m \leq 2 \) and \( 0.0 \leq \Omega_\Lambda \leq 2 \),

\[
\chi^2 = \sum_i \frac{(m_i^{data} - m_i^{th})^2}{\sigma_i^2}
\]

Here, \( m_i^{data} \) is the observational apparent magnitude of SN Ia data. \( m_i^{th} \) is the theoretically calculated apparent magnitude and \( \sigma_i \) is the observational uncertainty.
In the present work, we do not consider the data for the redshifts \( z < 0.01 \), because nearby SNIa are affected by peculiar motion [20]. As well it is worthwhile to consider the velocity uncertainty in modeling [20]. Therefore \( \sigma \) is modified as,

\[
\sigma^2_i = \sigma^2_{\text{data}} + \left( \frac{dm_{th} v}{dz c} \right)^2,
\]

where \( \sigma_{\text{data}} \) is the observational uncertainty of the apparent magnitude. Here \( c \) is the velocity of light and velocity uncertainty \( v = 500 \text{ km sec}^{-1} \) [20]. As well \( v/c \) is defined as the redshift error \( z_{\text{error}} = v/c \) [20]. Multiplying \( z_{\text{error}} \) by \( dm_{th}/dz \), uncertainty of the apparent magnitude is calculated.

We have done our calculation in the redshift range \( 0.01 \leq z \leq 2 \). We did the \( \chi^2 \) fitting to find the best parameter range for \( \Omega_m \) and \( \Omega_{\Lambda} \) with the SNIa observations. Figure 2.6 shows the contour results for confidence levels of 1\( \sigma \) for 68.3\%, 2\( \sigma \) for 95.4\% and 3\( \sigma \) for 99.73\% confidence levels. 1\( \sigma \) confidence level for \( \Omega_m \) is in the range of \( 0.15 < \Omega_m < 0.92 \) and \( \Omega_{\Lambda} \) is in the range of \( 0.38 < \Omega_{\Lambda} < 1.35 \), for \( M = -19.0 \) and \( H_0 = 76 \text{ km s}^{-1} \text{ Mpc}^{-1} \).

It is worthwhile to do a comparison with the reported results of CMB (Cosmic Microwave Background), the WMAP (Wilkinson Microwave Anisotropy probe) teams from a high angular resolution all-sky map of structure [21] . Their main conclusion is that we live in geometrically flat universe with \( \Omega_m \) and \( \Omega_{\Lambda} \) of \( 0.258 \pm 0.03 \) and \( 0.742 \pm 0.03 \), respectively. It is consistent with our results.

Figure 2.7 shows the illustration for \( \Omega_{\Lambda} \) vs \( \chi^2 \) for the flat universe where \( H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1} \) and \( M = -19.2 \). For 2\( \sigma \) confidence level \( \Omega_{\Lambda} \) is in the range of \( 0.68 < \Omega_{\Lambda} < 0.81 \). Our result is consistent with WMAP [21].

This is a good evidence for the existence of the cosmological term. However to solve the cosmological constant problem, we need new model with varying cosmological term beyond the standard model. From the next chapter we consider a one of non-standard models to work with this fine tuning problem.
Figure 2.6: Constrains in the $\Omega_M - \Omega_\Lambda$ plane. Dotted lines from middle to outward indicate the $1\sigma$, $2\sigma$, $3\sigma$ confidence levels for the fitting of the magnitude-redshift relation to the SNIa observations.
Figure 2.7: Illustration for the $\Omega_\Lambda$ vs $\chi^2$ for the flat universe; $M = -19.2, H_0 = 72$ [km s$^{-1}$ Mpc$^{-1}$]
3 Non-Standard Cosmology

To solve the cosmological constant problem, we can imagine the cosmological term may decrease from a large value at the early epoch to the present value. Therefore various functional form can be suggested. Among them we consider the Brand-Dicke (BD) model with a variable cosmological term $\Lambda$ as a function of the scalar field $\phi$.

### 3.1 Brans-Dicke model with a variable cosmological term $\Lambda$ as a function of the scalar field $\phi$ (BDA)

Cosmological term which is an explicit function of a scalar field $\phi$, was proposed by [22, 7]. The action in the Brans-Dicke theory modified with a variable cosmological term $\Lambda(\phi)$ is introduce by Endo and Fukui (1977) [8] as,

$$S = \int \sqrt{-g} L d^4x = \int d^4x \sqrt{-g} \left( (R - 2\Lambda) \phi - \frac{\omega}{\phi} \phi_i \phi^i + 16\pi \mathcal{L} \right),$$  \hfill (3.1)

where, $R$ is the scalar curvature, $\mathcal{L}$ is the Lagrangian density of matter which is assumed to depend explicitly on derivatives of $g_{\mu\nu}$. $\omega$ is a coupling constant and $\phi$ represents the scalar field and it plays a role analogous to $G^{-1}$.

The variational principle yields the metric field equation as,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2} \left( \phi_{\mu\nu} \phi_{\phi^k} - \frac{1}{2} g_{\mu\nu} \phi_{\phi^k} \phi_{\phi^k} \right) + \frac{1}{\phi} \left( \phi_{\mu\nu} - g_{\mu\nu} \nabla \phi \right),$$  \hfill (3.2)

where,

$$T^{\mu\nu} = \frac{2}{(-g)^{\frac{1}{2}}} \frac{\partial}{\partial g_{\mu\nu}} \left[ (-g)^{\frac{1}{2}} 2\mathcal{L} \right]$$  \hfill (3.3)
is the energy momentum tensor of matter.

Contraction of Eq. (3.2) results in,

\[-R - 4\Lambda = \frac{8\pi}{\phi} T_{\mu\nu} - \frac{\omega}{\phi^2} \phi_{\mu} \phi^\mu - \frac{3}{\phi} \Box \phi.\]  \hfill (3.4)

The field equation for \(\phi\) obtained by varying \(\phi\) and \(\phi_{,\mu}\) in Eq. (3.2) is as follows,

\[R - 2\Lambda - 2\phi \frac{\partial \Lambda}{\partial \phi} = \frac{\omega}{\phi^2} \phi_{,\mu} \phi^\mu - \frac{2\omega}{\phi} \Box \phi.\]  \hfill (3.5)

By eliminating \(R\) from Eq. (3.5) and Eq. (3.4), we can get,

\[\Lambda - \phi \frac{\partial \Lambda}{\partial \phi} = \frac{4\pi}{\phi} T_{\mu}^{\mu} - \frac{2\omega + 3}{2\phi} \Box \phi.\]  \hfill (3.6)

The simplest case of the coupling of the field is assumed as [23],

\[\Box \phi \equiv \phi_{,\mu}^{\rho} = \frac{8\pi}{2\omega + 3} \mu T_{\rho}^{\rho},\]  \hfill (3.7)

where the constant \(\mu\) shows how much this modified theory including \(\Lambda(\phi)\) deviates from that of the original Brans-Dicke theory.

The particular solution of Eq. (3.6) is given by

\[\Lambda = \frac{2\pi (1 - \mu)}{\phi} T_{\rho}^{\rho}.\]  \hfill (3.8)

It is noticed that the model with \(\mu = 1\) (\(\Lambda = 0\)) reduces to the original Brans-Dicke theory. Eq. (3.1) reduces to the action of General relativity when \(\phi = constant\).
3.2 Equation of motion

Substituting $R$ from Eq. (3.4) to Eq. (3.2), Eq. (3.2) can be rewritten as,

$$R_{\mu\nu} = g_{\mu\nu}\Lambda + \frac{8\pi}{\phi} \left[ T - \mu\nu + \frac{1}{2} \left( \frac{\mu}{2\omega + 3} - 1 \right) g_{\mu\nu}T^k_k \right] + \frac{\omega}{\phi^2}\phi_{\mu\nu\mu\nu} + \frac{1}{\phi}\phi_{\mu\nu} \tag{3.9}$$

$\Box\phi$ can be replaced by using Eq. (3.7). To solve Eq.(3.9), let us consider the metric for a general isotropic an homogeneous universe, Robertson-Walker metric as,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\psi^2) \right]$$

$$\tilde{g}_{tt} = -1 \tag{3.10}$$

$$\tilde{g}_{ij} = a^2(t) \tilde{g}_{ij} \tag{3.11}$$

$$\tilde{g}_{rr} = (1 - kr^2)^{-1}, \tilde{g}_{\theta\theta} = r^2, \tilde{g}_{\psi\psi} = r^2 \sin^2\theta$$

Here, $t$ is the cosmic time coordinate: $i$ and $j$ run over three comoving spatial coordinates $\theta$, $\psi$ and $\tilde{g}_{ij}$ is the metric for a three-dimensional maximally symmetric space: Here the velocity of light $c = 1$. In the homogeneous and isotropic universe, the distance between any two comoving points are proportional to a universal scale factor $a(t)$, at the cosmic time $t$.

Constant $k$ is related to the special geometry of the universe:

$$k = \begin{cases} 
1 & \text{close universe} \\
0 & \text{flat universe} \\
-1 & \text{open universe} 
\end{cases}$$

Elements of the Ricci tensor $R_{\mu\nu}$ can be written as follows,
\[ R_{tt} = \frac{3\ddot{a}}{a}, \quad R_{tt} = R_{ti} = 0, \quad R_{ij} = -(a\dddot{a} + 2a\dddot{a}^2 + 2k) \tilde{g}_{ij}, \]  

(3.12)

where \( R_{ij} \) is the spacial Ricci tensor calculated by the metric \( \tilde{g}_{ij} \).

Energy momentum tensor \( T_{\mu \nu} \) can be written in the form of perfect fluid,

\[ T_{\mu \nu} = p g_{\mu \nu} + \rho U_{\mu} U_{\nu}, \]  

(3.13)

where \( p \) is the pressure, \( \rho \) is the density and \( U_{\mu} \), the "Velocity four vectors", is given by \( U^{t} = 1, \ U^{i} = 0 \). Substituting \( g_{\mu \nu} \) and \( T_{\mu \nu} \) in (3.9) it can be rewritten as,

\[ R_{tt} = -\Lambda + \frac{8\pi}{\phi} \left[ \rho + \frac{1}{2} \left( \frac{\mu}{2\omega + 3} - 1 \right) (\rho - 3p) \right] + \omega \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\ddot{\phi}}{\phi}. \]  

(3.14)

Spacial part of the Ricci Tensor is written as,

\[ R_{ij} = a^2 \tilde{g}_{ij} \Lambda + \frac{8\pi}{\phi} \left[ \rho + \frac{1}{2} \left( \frac{\mu}{2\omega + 3} - 1 \right) (3p - \rho) \right] a^2 \tilde{g}_{ij} - \frac{\dot{\phi}}{\phi} \tilde{a} \tilde{g}_{ij}. \]  

(3.15)

Substituting from Eq. (3.12) to (3.14) and (3.15), we get

\[ \frac{3\ddot{a}}{a} = -\Lambda + \frac{8\pi}{\phi} \left[ \rho + \frac{1}{2} \left( \frac{\mu}{2\omega + 3} - 1 \right) (\rho - 3p) \right] + \omega \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\ddot{\phi}}{\phi}, \]  

(3.16)

\[ -(a\dddot{a} + 2a\dddot{a}^2 + 2k) = a^2 \dddot{g}_{ij} \Lambda + \frac{8\pi}{\phi} \left[ \rho + \frac{1}{2} \left( \frac{\mu}{2\omega + 3} - 1 \right) (3p - \rho) \right] a^2 - \frac{\dot{\phi}}{\phi} a \ddot{a}. \]  

(3.17)

Replacing \( \ddot{a} \) from Eq. (3.16) to (3.17),

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} + \frac{4\pi}{3\phi} \left[ 2\rho + \frac{\mu}{2\omega + 3} (3p - \rho) \right] a_2 + \frac{\omega}{6a} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\ddot{\phi}}{6a} \frac{\dot{\phi}}{2a}. \]  

(3.18)

Substituting the energy momentum tensor from Eq. (3.13) to the Eq. (3.7),

\[ \phi^{\mu}_{;\mu} = -\phi - 3\phi \frac{\dot{a}}{a} = \frac{8\pi \mu}{2\omega + 3} (3p - \rho) \]  

(3.19)

Substituting Eq. (3.19) in Eq. (3.18), the equation of motion is written as a differential equation in terms of \( a \) and \( \phi \).
\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} - \frac{\Lambda}{3} - \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\dot{\phi}}{a} \frac{\dot{\phi}}{\phi} = \frac{8\pi}{3\phi} \rho. \]  

(3.20)

As well from Eq. (3.19),

\[ \frac{d}{dt} \left( \frac{\dot{\phi}a^3}{\phi} \right) = -\frac{8\pi\mu}{2\omega + 3 (3p - \rho)} a^3. \]  

(3.21)

Eq. (3.20) and Eq. (3.21) are the equations of motions in the Brand-Dicke model with a variable \( \Lambda \) as a function of \( \phi \).

This model is an extension of the original BD theory and reduces to the Friedmann model when \( \phi = \text{constant}, \mu = 1, \) and \( \omega >> 1 \).
3.3 **BDE model and physical parameters**

3.3.1 Basic equations and physical quantities at the present

Curvature constant \( k \) can be find from Eq. (3.20) at \( t_0 = 0 \) as,

\[
k = \frac{8\pi}{3} \rho_0 - H_0^2 + \frac{\omega}{6} \left( \dot{\phi} \right)_0 - H_0 \left( \frac{\dot{\phi}}{\phi} \right)_0 + \frac{\Lambda}{3}.
\]  (3.22)

Here, \( H_0 \equiv (\dot{a}/a)_0 \) is the Hubble constant, \( \rho_0 \) is the present total density and \( a_0 = 1 \). Where the subscript ”0” indicates the present value. Gravitational ”constant” \( G \) in **BDE** model can be express as follows,

\[
G = \frac{1}{2} \left( 3 - \frac{2\omega + 1}{2\omega + 3} \mu \right) \frac{1}{\phi}.
\]  (3.23)

Gravitational constant \( G \) is varying with scalar field \( \phi \) and when \( \mu = 1 \), \( G \sim \phi^{-1} \).

Using Eq. (3.8) and energy momentum tensor Eq. (3.13),

\[
\Lambda = \frac{8\pi (\mu - 1)(\rho - 3p)}{\phi}.
\]  (3.24)
3.3.2 Equation of State

Using Eq. (3.13) for the energy conservation $T^\mu_{\nu} = 0$, defined the relation in between pressure $p$ and density $\rho$ as,

$$ \frac{d}{da} \left( \rho a^3 \right) + p d \left( a^3 \right) = 0 $$

(3.25)

If the universe is filled with perfect fluid with an equation of state,

$$ p = \alpha \rho, $$

(3.26)

where $\alpha$ is assumed to be constant. Substituting Eq. (3.26) in to Eq. (3.25),

$$ \frac{d}{dt} \left( \rho a^{3(\alpha+1)} \right) = 0, $$

(3.27)

and we can solve to give,

$$ \rho = \rho_0 a^{3(\alpha+1)}. $$

(3.28)

For the matter dominant universe, $\alpha = 0$ and $p_m \simeq 0$ then Eq. (3.28) can be written as,

$$ \rho_m a^3 = constant = \rho_m a_0^3 = \rho_{m0}. $$

(3.29)

Further more,

$$ \rho_m = constant = \rho_{m0} a^{-3}. $$

(3.30)

For the radiation dominant universe, $\alpha = 1/3$ we can get,

$$ \rho_r = \rho_{r0} a^{-4}. $$

(3.31)

Total pressure and total energy density is given as,

$$ \rho = \rho_m + \rho_r, \ p = p_m + p_r \simeq p_r = \frac{\rho_r}{3}. $$

(3.32)

We assume that, the pressure is satisfies $p = \rho/3$, which is legitimate only for relativistic particles. Then Eq. (3.21) is integrated to give,

$$ \dot{\phi} = \frac{1}{a^3} \left( \frac{8 \pi \mu}{2 \omega + 3} \rho_{m0} t + B \right) $$

(3.33)
where $B$ is an integral constant [23].

From Eq. (3.32) and Eq. (3.24), $\Lambda$ is rewritten as,

$$
\Lambda = \frac{2\pi (\mu - 1)}{\phi} \rho_m.
$$

(3.34)
3.4 Parameters

To solve the Eq. (3.20) and Eq. (3.21), we need some numerical values of the parameters that includes fundamental constants. The parameters that we use for numerical simulation are shown in the Table 3.1.

Table 3.1: Parameter values for the numerical calculation

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$: coupling constant</td>
<td>10000 [24]</td>
</tr>
<tr>
<td>$\mu$: Intrinsic parameter in BD model</td>
<td>$-2 \sim 2$ [10, 11]</td>
</tr>
<tr>
<td>$B^*$: integral constant</td>
<td>$-10 \sim 10$ [11]</td>
</tr>
<tr>
<td>$H_0$: Hubble constant</td>
<td>71 km s$^{-1}$ Mpc$^{-1}$ [25]</td>
</tr>
<tr>
<td>$G_0$: gravitational constant</td>
<td>$6.6726 \times 10^{-8}$ dyn cm$^2$ g$^{-2}$[10]</td>
</tr>
<tr>
<td>$T_{\gamma 0}$: present photon temperature</td>
<td>2.725 K [26]</td>
</tr>
</tbody>
</table>

Here, $B^*$ is the normalized value of $B$ is defined as, $B^* = B/(10^{-24} \text{g cm}^{-3})$.

It has been trying to constrain cosmological parameters like $\omega$ using high-accuracy microwave anisotropy, solar system gravity experiments and galaxy surveys [27] and so on. Arai et al (1987) [9] have done their calculation under the assumption of the characteristic parameter of $\omega = 6$ in the $BDA$ model. Another constrain, such as those from nucleosyntheses [5], is comparable but more model dependent; the most detailed analysis [12] gives the constraint of $\omega > 50$. One of the most robust constraint on $\omega$, that exceeds 500, has been derived from timing experiment using the Viking space probe [27]. Furthermore, [10] has investigated $BDA$ model with use of $\omega = 500$. It must be worthwhile to investigate $BDA$ model with the use of a new value of $\omega \geq 500$ (e.g. Will 1984) considering the observations of light elements and the observed constraint of the variation of $G$. From the Cassini spacecraft, it is obtained that present value of $\omega > 40000$ [24]. Therefore we use $\omega = 10000$ for our calculations to set the assumptions of our model.
Limits of the decreasing rate of $G$ has been found using several techniques like palaeomagnetica studies, masses of young and old neutron stars in pulsar binary limits, etc. Table 3.2 displays the each method used to obtain the upper limits of $(G_0/G)_0$. For following calculations we use the most severe observational limit as $(G_0/G)_0 < 10^{-12}$ yr$^{-1}$.

Table 3.2: Limitation of $(G/G)_0$

<table>
<thead>
<tr>
<th>Method</th>
<th>$(G/G)_0[yr^{-1}]$</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stability of radii of the Earth</td>
<td>$-(G/G)_0 \leq 8 \times 10^{-12}$</td>
<td>McElhinny et al. (1978)[27]</td>
</tr>
<tr>
<td>Big Bang Nucleosynthesis</td>
<td>$(G/G)_0 \leq 1.7 \times 10^{-13}$</td>
<td>Rothman et al. (1982) [28]</td>
</tr>
<tr>
<td>Lunar laser ranging</td>
<td>$(G/G)_0 \leq 8 \times (-2.0\pm10.0) \times 10^{-12}$</td>
<td>Shapiro(1990) [29]</td>
</tr>
<tr>
<td>Radar ranging of Mars</td>
<td>$(G/G)_0 \leq 8 \times 10^{-13}$</td>
<td>Müller et al. (1991) [30]</td>
</tr>
<tr>
<td>White dwarf</td>
<td>$(G/G)<em>0 \leq 8 \times (-3^{+11}</em>{-3}) \times 10^{-11}$</td>
<td>Garcia-Berro et al. (1995) [31]</td>
</tr>
<tr>
<td>Pulsar Binary limit</td>
<td>$(G/G)<em>0 \leq 8 \times (-0.6^{+2.0}</em>{-0}) \times 10^{-12}$</td>
<td>S.E.Thorsett(1996) [32]</td>
</tr>
<tr>
<td>Helioseismology</td>
<td>$</td>
<td>(G/G)_0</td>
</tr>
</tbody>
</table>

Note that $\Lambda < 0$ if $\mu < 1$, and $\phi G < 0$ if $\mu > 3$ and $\omega \gg 1$.

As can we see in Eq. (3.33), the behavior of the early universe is essentially determined by $B$.

$$\lim_{t \to 0} \frac{\dot{\phi}(t)}{B} = \lim_{t \to 0} \frac{B}{x^3(t)}$$

The model with negative $B$ implies negative $\dot{\phi}$, or equivalently positive $G$ in early universe.

Considering the present age of the universe, first term in the left side in Eq. (3.34) can be written as,

$$\frac{8\pi\mu}{2\omega + 3} \rho m_0 t_0 \sim 10^{-3} \times 10^{-31} \times 10^{17} = 10^{-17} \gg B,$$

where $t_0$ is the present age of the universe, $t_0 = 13.69 \pm 0.13$ Gyr $\sim 10^{17}$sec [21].

Then $B$ can be neglected from Eq. (3.33). Then using Eq. (3.23) and Eq. (3.33), an equation for the variation of $G$ is described as,
\[
\left( \frac{\dot{G}}{G} \right)_0 = - \left( \frac{\dot{\phi}}{\phi} \right)_0 = - \frac{16\pi \rho_m G_0 t_0}{3(2\omega + 3) - (2\omega + 1)\mu}.
\]

(3.35)

Note that \( \Lambda < 0 \) if \( \mu < 1 \), and \( \phi G < 0 \) if \( \mu > 3 \) and \( \omega \gg 1 \). If \( \mu = 3 \) the age of the universe becomes too short [10]. Considering these matter, \( \mu = 2 \) is taken as the upper limit of the parameter \( \mu \) for our calculations.
3.5 Flat universe in $BDA$ model

For the flat universe at the present epoch Eq. (3.20) is written as,

$$\frac{1}{3} \left( \frac{8\pi \rho_{m0}}{\phi_0} + \Lambda (\phi)_0 \right) + \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)_0^2 - \left( \frac{\dot{\phi}}{\phi} H \right)_0 = H_0^2. \quad (3.36)$$

For this model we use $\Lambda$ as $\Lambda(\phi)$ for the simplicity. Where $\Lambda(\phi)$ is a variable cosmological term as a function of the scalar field $\phi$.

Let us consider the order of the four terms in the left hand side.

Since $\phi_0 \propto G_0^{-1}$, $\phi_0 \simeq 10^7 \sim 10^8$ and $H_0 = h \text{sec}^{-1}/3.08568 \times 10^{17} = 2.300951 \times 10^{-18} \text{sec}^{-1}$ for $h = 0.71$, we get

$$\frac{\rho_{m0}}{\phi_0} \simeq 10^{-30} \times (10^{-7} \sim 10^{-8}) = 10^{-37} \sim 10^{-38} \text{s}^{-2}.$$

From Eq. (3.34),

$$\Lambda (\phi) \simeq 10^{-37} \sim 10^{-38} \text{s}^{-2}.$$

In Eq. (3.35), we estimate

$$\left( \frac{\dot{\phi}}{\phi} \right)_0^2 \omega \simeq (10^{-30} \times 10^{-8} \times 10^{17})^2 \times 10^4 = 10^{-38} \text{s}^{-2},$$

$$\left( \frac{\dot{\phi}}{\phi} H \right)_0 \simeq 10^{-21} \times 10^{-18} \text{s}^{-2} = 10^{-39} \text{s}^{-2}.$$

For right hand side term, $H_0^2 \simeq 10^{-36} \text{s}^{-2}$.

Then comparing the four terms in the left hand side, the first two terms are dominant than the third and fourth terms. Therefore third and fourth terms may be neglected.

Then the Eq. (3.36) is simplified as,

$$\frac{1}{3} \left( \frac{8\pi \rho_{m0}}{\phi_0} + \Lambda (\phi)_0 \right) = H_0^2, \quad (3.37)$$

$$\frac{8\pi \rho_{m0}}{3\phi_0 H_0^2} + \frac{\Lambda (\phi)_0}{3H_0^2} = 1. \quad (3.38)$$
Critical density for $\Lambda$ model is defined for a given Hubble parameter, in the absence of the cosmological term in the flat universe as,

$$\rho_c^{\Lambda} = \frac{3\phi_0 H_0^2}{8\pi},$$

(3.39)

where $\phi_0$ from Eq. (3.23) is written as,

$$\phi_0 = \frac{1}{G_0} \left(3 - \frac{2\omega + 1}{2\omega + 3\mu}\right).$$

(3.40)

This implies that $\rho_c^{\Lambda}$ depends on $\mu$. Eq. (3.38) is rewritten as,

$$\frac{\Lambda (\phi)_0}{3H_0^2} = \frac{(\mu - 1) \rho_{m0}}{4\rho_c^{\Lambda}},$$

(3.41)

$$\frac{\rho_{m0}}{\rho_c^{\Lambda}} + \frac{(\mu - 1) \rho_{m0}}{4\rho_c^{\Lambda}} = 1.$$  

(3.42)

Let us define the density parameter as,

$$\Omega_{m0} = \frac{\rho_{m0}}{\rho_c^{\Lambda}},$$

(3.43)

$$\lambda_0 = \frac{(\mu - 1) \rho_{m0}}{4\rho_c^{\Lambda}}.$$  

(3.44)

Then Eq. (3.42) is rewritten as,

$$\Omega_{m0} + \lambda_0 = 1.$$  

(3.45)

Using Eq. (3.42), $\rho_{m0}$ is defined in terms of $\rho_c^{\Lambda}$ and $\mu$ as,

$$\rho_{m0} = \frac{4\rho_c^{\Lambda}}{(\mu + 3)}.$$  

(3.46)
3.6 Flat universe in $BDA$ with a constant cosmological term.

As the next approach $\Lambda(\phi)$ is modified by adding a constant cosmological term $\Lambda_{c0}$. Eq. (3.37) can be defined with the constant cosmological term $\Lambda_{c0}$.

\[
\frac{1}{3} \left( \frac{8\pi \rho_0}{\phi_0} + \Lambda (\phi)_0 + \Lambda_{c0} \right) = H_0^2
\]  

(3.47)

Same as the Eq. (3.38), above equation also can be defined with $\Lambda_{c0}$ as,

\[
\frac{8\pi \rho_0}{3\phi_0 H_0^2} + \frac{\Lambda (\phi)_0}{3H_0^2} + \frac{\Lambda_{c0}}{3H_0^2} = 1.
\]  

(3.48)

Defining $\Lambda_{c0}/3H_0^2$ as $\lambda_{c0}$, Eq. (3.48) is rewritten as,

\[
\Omega_{m0} + \lambda_0 + \lambda_{c0} = 1.
\]  

(3.49)

Then $\rho_{m0}$ is defined as,

\[
\rho_{m0} = \frac{4(1 - \lambda_{c0}) \rho_c^{BDA}}{(\mu + 3)}.
\]  

(3.50)
3.7 Computational results of \( BDA \) for flat universe

Computational results of \( BDA \) model in the flat universe for several values of \( B^* \) are shown in the Figure 3.1 and Figure 3.2 for \( \omega = 10000 \), \( h = 0.71 \). Figure 3.1 shows the evolution of the scale factor with time. It is noted that the solution with \( B^* = 0 \) behaves very similar to the Friedman model in the early universe. As well the evolutionary path for \( B^* \neq 0 \) deviates appreciable in the early universe. It has a difference of expansion rates at \( t < 10 \text{ s} \) with \( B^* \). As well when \( |B^*| \) increases, expansion rate also increases at \( t < 10 - 100 \text{ s} \). This is caused of the behavior of Eq. (3.33) at early universe. At the early universe \( B \) is very effective to determine the scalar field.

Figure 3.2 shows the time variation of \( G \) for several values of \( B^* \). The model with \( B^* = 0 \) reaches to nearly constant \( G \) in the early universe, similar to the Friedman universe. As can be seen in Eq. (3.23) the behavior of \( \dot{\phi} \) is essentially determined by the value of \( B \) in the early universe. It is conventional to consider \( \phi \) as an increasing function of time for positive \( B \) and it implies that \( G \) is an decreasing function with time for \( B > 0 \) in the early universe. As well for negative \( B \), \( \dot{\phi} \) also negative in the early universe. But it changes to positive some later stage of the universe. As well it can be noticed that dependence on \( B^* \) is insensitive for \( t > 8 \text{ s} \).

Figure 3.3 shows the evolution of the Hubble parameter with the scale factor with changing \( \mu \) and fixing \( B^* = -10 \) for \( BDA \) model. It is noticed that the dependence of the Hubble parameter on \( \mu \) is insensitive when the scale factor \( a \sim 10^{-4} \).

Figure 3.4 explains the evolution of the Hubble parameter with the scale factor by changing \( B^* \) and fixing \( \mu = 0.7 \) in the \( BDA \) model. It can be noticed that, \( B^* \) is not effective for the evolution of the Hubble parameter after scale factor reaches to \( a \sim 10^{-8} \). We have discussed the effectiveness of \( B \) in the early universe for the the scale factor in the description of Figure 3.1.
Figure 3.1: Evolution of scale factors as a function of time in BDA model for \( \mu = 0.7 \) with \( B^* \) changed. Dotted line with ”FRW” indicates the Friedmann universe.
Figure 3.2: Time variation of the gravitational "constant" $G$ with $B^*$ changed and $\mu = 0.7$ fixed, for the flat universe in the $BDA$ model
Figure 3.3: Variation of the Hubble parameter $H[\text{sec}^{-1}]$ with the scale factor $a$ with $B^*$ changed and $\mu = 0.7$ fixed, for the flat universe in the $BDA$ model with $\Lambda(\phi)$
Figure 3.4: Variation of the Hubble parameter $H[\text{sec}^{-1}]$ with scale factor $a$ with $\mu$ changed under the fixed value of $B^* = -10$ for the flat universe with the variable cosmological term in the $BD\Lambda$ model.
4 Magnitude-redshift relation in a non-standard model

4.1 Formula of the magnitude-redshift relation.

In chapter 2 magnitude-redshift \((m - z)\) relation has been investigated to the Friedmann model. Here we apply \(m - z\) relation to the \(BDA\) model. We note that this \(\Lambda\) includes variable and constant cosmological terms. Here we use Eq. (2.9) which has been derived in Chapter 2 as,

\[
\int_{0}^{z} \frac{dz}{H} = \int_{0}^{r} \frac{dr}{\sqrt{(1 - kr^2)}}. \tag{4.1}
\]

From Eq. (3.20),

\[
\left(\frac{\dot{a}}{a}\right)^2 + k \frac{a^2}{a^2} - \frac{\rho}{3} - \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\dot{\phi}}{a} \frac{\phi}{a} = \frac{8\pi}{3} \rho. \tag{4.2}
\]

This is a quadratic equation of \(\dot{a}/a\), where \(H = \dot{a}/a\):

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{\phi}}{\phi} \left(\frac{\dot{a}}{a}\right) + \left(\frac{k}{a^2} - \frac{\lambda}{3} - \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{8\pi}{3} \rho\right) = 0 \tag{4.3}
\]

Solving the above equation, the solutions for the Hubble parameter \(H\) is found as,

\[
H = \pm \left[ \frac{1}{4} \left(\frac{\dot{\phi}}{\phi}\right)^2 - (1 + z)^2 k - \frac{\lambda}{3} - \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{8\pi}{3} \rho \right]^{\frac{1}{2}} - \frac{1}{2} \frac{\dot{\phi}}{\phi}. \tag{4.4}
\]

Since the universe is expanding, we take the positive solution. Then substituting the Hubble parameter in Eq. (4.1),
\[ \int_0^{r_l} \frac{dr}{\sqrt{1 - kr^2}} = \int_0^z \frac{dz}{\left[ \frac{1}{4} \left( \frac{\dot{\phi}}{\phi} \right)^2 - (1 + z)^2 k - \frac{\Lambda}{3} - \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{8\pi}{3\phi} \rho \right]^{\frac{1}{2}} - \frac{1}{2} \frac{\dot{\phi}}{\phi}}. \]  

(4.5)

For the simplicity, we take \( \zeta \) as,

\[ \zeta = \int_0^z \frac{dz}{\left[ \frac{1}{4} \left( \frac{\dot{\phi}}{\phi} \right)^2 - (1 + z)^2 k - \frac{\Lambda}{3} - \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{8\pi}{3\phi} \rho \right]^{\frac{1}{2}} - \frac{1}{2} \frac{\dot{\phi}}{\phi}}. \]  

(4.6)

Constant \( k \) is related with the spacial geometry of the universe.

When the universe is flat \((k = 0)\), Eq. (4.5) becomes,

\[ \int_0^{r_l} \frac{dr}{\sqrt{1 - kr^2}} = r_l, \]  

(4.7)

\[ r_l = \zeta, \]  

(4.8)

\[ r_l = \int_0^z \frac{dz}{\left[ \frac{1}{4} \left( \frac{\dot{\phi}}{\phi} \right)^2 - (1 + z)^2 k - \frac{\Lambda}{3} - \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{8\pi}{3\phi} \rho \right]^{\frac{1}{2}} - \frac{1}{2} \frac{\dot{\phi}}{\phi}}. \]  

(4.9)

For the close universe, where \( k > 0 \),

\[ r_l = \frac{1}{\sqrt{k}} \sin \sqrt{k} \zeta. \]  

(4.10)

For the open universe, where \( k < 0 \),

\[ r_l = \frac{1}{\sqrt{|k|}} \sinh \sqrt{|k|} \zeta. \]  

(4.11)

Apparent magnitude \( m \) is defined in Chapter 2, in Eq. (2.5) as,

\[ m = 5 \log \left( \frac{(1 + z) r_l}{10 \text{pc}} \right) + M. \]  

(4.12)

Applying \( r_l \) to the above equation, apparent magnitude can be calculated for the Brans-Dicke models with both variable and constant cosmological terms.
4.2 Comparison with SNIa observations for flat universe

4.2.1 Case for the BDΛ

For this model Hubble parameter is defined by Eq.(4.3) as,

\[
H = \pm \left[ \frac{1}{4} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{\Lambda(\phi)}{3} - \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{8\pi}{3\phi} \rho \right]^{\frac{1}{2}} - \frac{1}{2} \frac{\dot{\phi}}{\phi} \tag{4.13}
\]

Applying \( H \) into Eq. (2.8) \( r_l \) can be derived and then apparent magnitude also can be derived. In this section we investigate the \( m - z \) relation in \( BDΛ \) model, specially for flat universe.

First we consider the \( BDΛ \) model. As describes in Eq. (3.46), present matter density \( \rho_{m0} \) is defined. From Eq. (3.41) it is noticed that the present value of the energy density of \( \Lambda(\phi) \) depends on \( \mu \) and \( \rho_{m0} \).

Using Eq. (3.43) and Eq. (3.44) \( \Omega_{m0} \) and \( \lambda_0 \) are calculated. When \( \mu < 1 \) then \( \lambda_0 \) represents the negative energy. The maximum positive value for \( \lambda_0 = 0.2 \) can be taken for \( \mu = 2 \). Whatever the value we use for \( \mu \) in the range of \(-2.0 < \mu < 2.0\), we always get the matter dominant universe. Then it is noticed that, for several values of \( \mu \) we get the same \( m - z \) relation. To check the consistency with the SNIa observations we consider the redshift in the range of \( 0.01 < z < 2 \). As shown in the Figure 3.3 and Figure 3.4 it is seen that the Hubble parameter is independent on \( \mu \) and \( B \) when \( a \sim 10^{-4} \). Therefore, for the redshift range \( 0.01 < z < 2 \), we get the parameter independent relation. We get the \( \chi^2 \) value of 416 for this model. This model is inconsistent with the SNIa observations in the high redshifts as shown in the Figure 4.1.

Next to obtain the matter, radiation and cosmological term contributions through the evolution of the universe, we define matter, radiation and cosmological term densities as \( \rho_m, \rho_r, \) and \( \rho_\Lambda; \)

\[
\rho_\Lambda = \frac{(\mu - 1) \rho_m}{4}. \tag{4.14}
\]
Figure 4.1: Constrains from SNIa observations in $BDA$ model with $\mu = 0.7$ and $B^* = -10$
Figure 4.2 shows that the relation of these three densities with the time in flat $BDA$ model. This figure shows that the early universe is dominant by the radiation and matter is dominant at the present. It is noticed that evolution of the energy densities of cosmological term and matter are parallel to each other though $\rho_m$ is more dominant than $\rho_\Lambda$ for all the period. If the universe is dominant by the matter, universe is decelerating. Since $\rho_\Lambda$ is written in terms of $\rho_m$, these terms have the similar evolution.

Figure 4.6 shows that $m-z$ relation in the $BDA$ model is merged with the Friedmann model for $(\Omega_m, \Omega_\Lambda) = (1.0, 0.0)$.

Because of the inconsistency of this model with SNIa observations it is worthwhile to modify the cosmological term and check the consistency with SNIa observations.
Figure 4.2: Time evolution of the energy densities of the matter $\rho_m$, radiation $\rho_r$, and cosmological term $\rho_\Lambda$ in the flat $BD\Lambda$ model for $\mu = 0.7$ and $B^* = 5$. 
4.2.2 Case for the $BDA$ model with a constant cosmological term

Since $BDA$ model is inconsistent with SNIa observations, in this section we modify $\Lambda(\phi)$ by adding constant cosmological term $\Lambda_0$ as described in the section 3.6.

Hubble parameter in Eq.(4.3) is modified with the $\Lambda_0$ as,

$$H = \pm \left[ \frac{1}{4} \left( \frac{\dot{\phi}}{\phi} \right)^2 + \frac{\Lambda(\phi)}{3} + \frac{\Lambda_0}{3} - \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{8\pi}{3\phi} \rho \right]^{1/2} - \frac{1}{2} \frac{\dot{\phi}}{\phi}. \quad (4.15)$$

Applying $H$ into Eq. (2.8) $r_l$ can be derived and then apparent magnitude also derived. In this section we investigate the $m-z$ relation in $BDA$ model with constant cosmological term. For this simulation we fix the value of $\lambda_0$ as 0.7.

Same as the above model, we investigate the time evolution of the each density term for matter, radiation and cosmological terms ($\rho_\Lambda + \rho_{\Lambda_0}$). Figure 4.4 shows the evolution of the densities with the time for $\mu = 0.7$ and $B^* = 5$. It is noticed that at the present epoch cosmological term is dominant compared to the matter. Therefore we expect a accelerating universe, because there is a source to accelerate the universe. It can be proved from the Figure 4.5. Furthermore Figure 4.6 shows that the $m-z$ relation for the Friedmann model with $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ is merged with the $BDA$ model with constant cosmological term, with $\chi^2 = 196$. Therefore this model is consistent with SNIa observations and present accelerating universe.
Figure 4.3: Magnitude-redshift \((m - z)\) relation in the \(BDA\) model with and without constant cosmological term in the flat universe for, \(\mu = 0.7\) and \(B' = -10\)
Figure 4.4: Time evolution of the energy densities of the matter $\rho_m$, radiation $\rho_r$, and the total cosmological term $\rho_\Lambda = \rho_\Lambda^0 + \rho_\Lambda$ in the flat $BD\Lambda$ model with the constant cosmological term for $\mu = 0.7$ and $B^* = 5$. 
Figure 4.5: Magnitude-redshift relation for the flat universe in the $BDA$ model with and without constant cosmological term for $B^* = -10$ and $\mu = 0.7$
Figure 4.6: Magnitude-redshift relation for the Friedmann model and flat universe in the \textit{BDA} model with and without constant cosmological term for $B^* = -10$ and $\mu = 0.7$. 
5 Summary and Discussion

We have investigated the magnitude-redshift relation in the Brans-Dicke model with both variable and constant cosmological terms.

Before we consider the Brans-Dicke cosmology, magnitude-redshift relation has been investigated in the Friedmann model. Cosmological constant with the Friedmann model is tightly constrained from the SNIa observations with the minimum value of $\chi^2 = 194$. For the flat universe, we show the value range for $\Omega$ with the $\chi^2$ fitting. We found that the value of $\Omega$ is in the range of $0.68 < \Omega < 0.81$, which is consistent with WMAP results [21]. Though this is a good evidence to the existence of the cosmological constant, to find the solution for the cosmological constant problem, more functional forms of cosmological terms need to be suggested. Then we consider the Brans-Dicke model with the cosmological term varying as a function of a scalar field ($BD\Lambda$).

Though it has already been investigated from the point of the Big Ban nucleosynthesis [11, 9], we have needed to investigate furthermore the epoch with SNIa observations. Therefore as the first step we considered the flat universe in the BD model with the cosmological term which is varying with the scalar field. For this model, it has been found that contribution from the energy density of matter is larger than the energy density of the cosmological term. Specially for the parameter region $-0.7 < \mu < 2$, contribution from $\Omega_{\Lambda(\phi)}$ to the total density is always less than 30%.

When we have compared with the Fridmann model, the matter dominant universe with the parameters $(\Omega_{\Lambda}, \Omega_m) = (1.0, 0.0)$ is merged with this $BDA$ model with $\chi^2 = 416$. As well it is seen that, this model is inconsistent with SNIa observations and the present accelerating universe too. Cosmic distance measures sensitively depends on the present
densities of the various energy components. This model does not contain the fair amount of the source which is caused to acceleration of the universe. The universe does not accelerate in this model and therefore magnitude-redshift relation can not be constrained. Then some modification of the cosmological term should be required.

As the next approach variable cosmological term has been modified by adding a constant cosmological term by fixing the energy density parameter of the constant cosmological term as 0.7. In this model, the contribution from the energy density of cosmological term to the total energy density is larger than the matter energy density contribution. We can define the energy density of \( \Lambda(\phi) \) in terms of the present matter density. Therefore whatever the value we used for \( \mu \), \( \Omega_m + \Omega_{\Lambda(\phi)} \) gets the value 0.3 and then remaining 0.7 contributes from the constant cosmological term. Variable cosmological term is proportional to the matter density in our model. For \( \mu = 2, B = -10 \) we find \( \Omega_\Lambda = 6.0 \times 10^{-2} \). It is the same order to the present baryon density \( 0.0441 \pm 0.0030 \) [28].

For the above model, present universe is dominant by the cosmological term and consistent with SNIa observations with \( \chi^2 = 196 \). Therefore it is consistent with present accelerating universe. As well comparing with the Friedmann model, we can say that the accelerating universe has the same density parameters as Friedmann model where \( (\Omega_m, \Omega_\Lambda) = (0.3, 0.7) \). This is because the \( BDA \) model with the constant cosmological term has been converge to the Friedmann model in around \( (\Omega_m, \Omega_\Lambda) = (0.3, 0.7) \) parameter region.

The models whose parameters are inherent in the \( BDA \) model become independent as far as the magnitude-redshift relation at the present epoch is concerned. In other words, these parameters have been obtained from the Big Bang nucleosynthesis for the early universe time, we can not constrain these parameters around the present epoch using magnitude-redshift relation.

In the present study we just added a constant cosmological term mathematically consistent. Therefore it is worthwhile to introduce another functional forms of cosmological term. I am expecting to continue my future research with the Brans-Dicke model by introducing more general functional forms to the cosmological term. To understand the role
of the cosmological term in the present accelerating universe, magnitude-redshift relation is not enough because it is seriously depends on various energy densities and we can not distinguish each energy densities. To solve the cosmological constant problem, we find that the magnitude-redshift relation is not enough and we need further investigations from Big Bang nucleosynthesis, cosmic microwave background constraint, and general scalar tensor theories.
Acknowledgment

I should be grateful to Prof. Masa-aki Hashimoto, for giving me a opportunity to be a student in the theoretical Astrophysics group in Kyushu University. As well I am greatly indebted to him for his valuable advice and guidance to continue my research according to my research interest.

My special thank should go to Dr. Riou Nakamura who enlighten me on very valuable discussion, comments and it helped me to broaden my knowledge of cosmology and computer programming.

My gratitude’s are due to our laboratory members helping me and encourage me to handle the life in Japan. I should not forget Mr. M. Saruwathari, my personal tutor, who always help me in the Japanese language difficulties even in the midst of his busy schedules.

I should be thankful to "Khonan Asia Foundation”, who encourage me giving a scholarship in year 2008.

I should be grateful to the professors and Senior Lecturers in the Departments, of Mathematics and Physics in University of Kelaniya, Sri Lanka, of encouraging me to continue my studies through this field.

I should give my heartiest thank to my family members, relatives and friends who always in behind me as a shadow.
References


